

# Isometric copies of $l^\infty$ in Cesàro-Orlicz function spaces

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**ABSTRACT.** We characterize Cesàro-Orlicz function spaces  $Ces_\varphi$  containing order isomorphically isometric copy of  $l^\infty$ . We discuss also some useful applicable conditions sufficient for the existence of such a copy.

2010 Mathematics Subject Classification. 46A80, 46B04, 46B20, 46B42, 46E30.

Key words and phrases. Cesàro-Orlicz function space; isometric copy of  $l^\infty$ ; order continuous norm.

## 1. INTRODUCTION

The structure of different types of Cesàro spaces has been widely investigated during the last decades from the isomorphic as well as isometric point of view. The spaces generated by the Cesàro operator (including abstract Cesàro spaces) have been considered by Curbera, Ricker and Leśnik, Maligranda in several papers (see [12], [13], [14], [15], [26], [27], [28]). The classical Cesàro sequence  $ces_p$  and function  $Ces_p$  spaces have been studied by many authors (see [2], [3] - also for further references, [1], [4], [10], [11]). It has been proved among others that some properties are fulfilled in the sequence case and are not in function case. Moreover, sometimes the cases  $Ces_p[0, 1]$  and  $Ces_p[0, \infty)$  are essentially different (see an isomorphic description of the Köthe dual of Cesàro spaces in [2] and [26]).

The Cesàro-Orlicz sequence spaces denoted by  $ces_\varphi$  are generalization of the Cesàro sequence spaces  $ces_p$ . Of course, the structure of the spaces  $ces_\varphi$  is richer than of the space  $ces_p$ . The spaces  $ces_\varphi$  have been studied intensively (see [9], [19], [22] and [34]). We want to investigate the Cesàro-Orlicz function space  $Ces_\varphi(I)$ . The spaces  $Ces_\varphi$  which contain an order isomorphic copy of  $l^\infty$  (equivalently, are not order continuous) have been characterized in [21]. The monotonicity properties have been also considered in [21]. In this paper we want to describe Cesàro-Orlicz function spaces  $Ces_\varphi$  containing order isomorphically isometric copy of  $l^\infty$ . We discuss also some useful applicable conditions sufficient for the existence of such a copy. We admit the largest possible class of Orlicz functions giving the maximal generality of spaces under consideration.

## 2. PRELIMINARIES

Let  $\mathbb{R}$ ,  $\mathbb{R}_+$  and  $\mathbb{N}$  be the sets of real, nonnegative real and natural numbers, respectively. Denote by  $\mu$  the Lebesgue measure on  $I$  and by  $L^0 = L^0(I)$  the space of all classes of real-valued Lebesgue measurable functions defined on  $I$ , where  $I = [0, 1]$  or  $I = [0, \infty)$ .

A Banach space  $E = (E, \|\cdot\|)$  is said to be a Banach ideal space on  $I$  if  $E$  is a linear subspace of  $L^0(I)$  and satisfies two conditions:

- (i) if  $g \in E$ ,  $f \in L^0$  and  $|f| \leq |g|$  a.e. on  $I$  then  $f \in E$  and  $\|f\| \leq \|g\|$ ,
- (ii) there is an element  $f \in E$  that is positive on whole  $I$ .

Sometimes we write  $\|\cdot\|_E$  to be sure in which space the norm has been taken.

For two Banach ideal spaces  $E$  and  $F$  on  $I$  the symbol  $E \hookrightarrow F$  means that the embedding  $E \subset F$  is continuous, i.e., there exists constant a  $C > 0$  such that  $\|x\|_F \leq C\|x\|_E$  for all  $x \in E$ . Moreover,  $E = F$  means that the spaces are the same as the sets and the norms are equivalent.

A Banach ideal space is called order continuous ( $E \in (\text{OC})$  shortly) if every element of  $E$  is order continuous, that is, for each  $f \in E$  and for each sequence  $(f_n) \subset E$  satisfying  $0 \leq f_n \leq |f|$  and  $f_n \rightarrow 0$  a.e. on  $I$ , we have  $\|f_n\| \rightarrow 0$ . By  $E_a$  we denote the subspace of all order continuous elements of  $E$ . It is worth to notice that in case of Banach ideal spaces on  $I$ ,  $x \in E_a$  if and only if  $\|x\chi_{A_n}\| \rightarrow 0$  for any decreasing sequence of Lebesgue measurable sets  $A_n \subset I$  with empty intersection (see [5, Proposition 3.5, p. 15]).

A function  $\varphi : [0, \infty) \rightarrow [0, \infty]$  is called an Orlicz function if:

- (i)  $\varphi$  is convex,
- (ii)  $\varphi(0) = 0$ ,
- (iii)  $\varphi$  is neither identically equal to zero nor infinity on  $(0, \infty)$ ,
- (iv)  $\varphi$  is left continuous on  $(0, \infty)$ , i.e.,  $\lim_{u \rightarrow b_\varphi^-} \varphi(u) = \varphi(b_\varphi)$  if  $b_\varphi < \infty$ , where

$$b_\varphi = \sup\{u > 0 : \varphi(u) < \infty\}.$$

For more information about Orlicz functions see [8] and [24].

If we denote

$$a_\varphi = \sup\{u \geq 0 : \varphi(u) = 0\},$$

then  $0 \leq a_\varphi \leq b_\varphi \leq \infty$ . Moreover,  $a_\varphi < \infty$  and  $b_\varphi > 0$ , since an Orlicz function is neither identically equal to zero nor infinity on  $(0, \infty)$ . The function  $\varphi$  is continuous and non-decreasing on  $[0, b_\varphi)$  and is strictly increasing on  $[a_\varphi, b_\varphi)$ . We use notations  $\varphi > 0$ ,  $\varphi < \infty$  when  $a_\varphi = 0$ ,  $b_\varphi = \infty$ , respectively.

We say an Orlicz function  $\varphi$  satisfies the condition  $\Delta_2$  for large arguments ( $\varphi \in \Delta_2(\infty)$  for short) if there exists  $K > 0$  and  $u_0 > 0$  such that  $\varphi(u_0) < \infty$  and

$$\varphi(2u) \leq K\varphi(u)$$

for all  $u \in [u_0, \infty)$ . Similarly, we can define the condition  $\Delta_2$  for small, with  $\varphi(u_0) > 0$  ( $\varphi \in \Delta_2(0)$ ) or for all arguments ( $\varphi \in \Delta_2(\mathbb{R}_+)$ ). These conditions play a crucial role in the theory of Orlicz spaces, see [8], [24], [33] and [36]. We will write  $\varphi \in \Delta_2$  in two cases:  $\varphi \in \Delta_2(\infty)$  if  $I = [0, 1]$  and  $\varphi \in \Delta_2(\mathbb{R}_+)$  if  $I = [0, \infty)$ .

The Orlicz function space  $L^\varphi = L^\varphi(I)$  generated by an Orlicz function  $\varphi$  is defined by

$$L^\varphi = \{f \in L^0(I) : I_\varphi(f/\lambda) < \infty \text{ for some } \lambda = \lambda(f) > 0\},$$

where  $I_\varphi(f) = \int_I \varphi(|f(t)|) dt$  is a convex modular (for the theory of Orlicz spaces and modular spaces see [33] and [36]). The space  $L^\varphi$  is a Banach ideal space with the Luxemburg-Nakano norm

$$\|f\|_\varphi = \inf\{\lambda > 0 : I_\varphi(f/\lambda) \leq 1\}.$$

It is well known that  $\|f\|_\varphi \leq 1$  if and only if  $I_\varphi(f) \leq 1$ . Moreover, the set

$$(2.1) \quad KL^\varphi = KL^\varphi(I) = \{f \in L^0(I) : I_\varphi(f) < \infty\},$$

will be called the Orlicz class.

The Cesàro operator  $C : L^0(I) \rightarrow L^0(I)$  is defined by

$$Cf(x) = \frac{1}{x} \int_0^x f(t)dt,$$

for  $0 < x \in I$ . For a Banach ideal space  $X$  on  $I$  we define an abstract Cesàro space  $CX = CX(I)$  by

$$CX = \{f \in L^0(I) : C|f| \in X\}$$

with the norm  $\|f\|_{CX} = \|C|f|\|_X$  (see [26], [27], [28]).

The Cesàro-Orlicz function space  $Ces_\varphi = Ces_\varphi(I)$  is defined by  $Ces_\varphi(I) = CL^\varphi(I)$ . Consequently, the norm in the space  $Ces_\varphi$  is given by the formula

$$\|f\|_{Ces(\varphi)} = \inf\{\lambda > 0 : \rho_\varphi(f/\lambda) \leq 1\},$$

where  $\rho_\varphi(f) = I_\varphi(C|f|)$  is a convex modular. We always assume that  $Ces_\varphi \neq \{0\}$ . If  $I = [0, \infty)$  then  $Ces_\varphi[0, \infty) \neq \{0\}$  if and only if the function  $x \rightarrow \frac{1}{x}\chi_{[a, \infty)}(x), x > 0$  belongs to  $L^\varphi[0, \infty)$  for some  $a > 0$  (see Theorem 1 (a) in [26] and Proposition 3 in [21]). However,  $Ces_\varphi[0, 1] \neq \{0\}$  for any Orlicz function  $\varphi$ . Indeed,  $L^\varphi[0, 1]$  is symmetric and  $Ces_\varphi[0, 1] \neq \{0\}$  if and only if  $\chi_{[a, 1]} \in L^\varphi[0, 1]$  for some  $0 < a < 1$  (see Theorem 1 (b) in [26]).

In this paper we accept the convention that  $\sum_{n=m}^k x_n = 0$  if  $k < m$ .

Note that if  $0 < a_\varphi = b_\varphi$  then  $L^\varphi = L^\infty$  and  $\|x\|_\varphi = \frac{1}{b_\varphi} \|x\|_\infty$ , see Example 1 in [33, p. 98]. Consequently,  $Ces_\varphi = Ces_\infty$  in that case (see [2], [26]). Therefore we can assume that if  $b_\varphi < \infty$ , then  $a_\varphi < b_\varphi$ .

### 3. ISOMETRIC COPIES OF $l^\infty$ IN $Ces_\varphi$

Define a set

$$C_\varphi = C_\varphi(I) = \{x \in Ces_\varphi(I) : \rho_\varphi(kx) < \infty \text{ for all } k > 0\}.$$

Note that  $C_\varphi = \{0\}$  whenever  $b_\varphi < \infty$ . If  $b_\varphi = \infty$ , then  $(Ces_\varphi)_a = C_\varphi$  by Theorem 5 from [21] (recall that  $(Ces_\varphi)_a$  is the subspace of all order continuous elements in  $Ces_\varphi$ ).

We say that a measurable set  $\Omega$  is a support of a Banach ideal space  $E$ , we write  $\text{supp } E = \Omega$  whenever

- (i) for each  $x \in E$  there is a measurable set  $A$  with  $\mu(A) = 0$  and  $\text{supp } x \subset A \cup \Omega$ .
- (ii) there is  $x \in E$  such that  $\mu(\Omega \setminus \text{supp } x) = 0$ .

**Lemma 1.** Suppose  $b_\varphi < \infty$ . Then  $(L^\varphi)_a = \{0\}$  and  $\text{supp } (Ces_\varphi)_a = I$ . Moreover, if  $0 \leq x \in (Ces_\varphi)_a$ , then  $\lim_{t \rightarrow 0^+} Cx(t) = 0$ .

*Proof.* The equality  $(L^\varphi)_a = \{0\}$  is well known, it is enough to consider the element  $x = \alpha\chi_A$  for  $\alpha > 0$  and  $0 < \mu(A) < \infty$ . Taking a sequence  $(A_n)$  of measurable subsets of  $A$  with  $\mu(A_n) \rightarrow 0$  we conclude that  $\|x\chi_{A_n}\|_\varphi \nrightarrow 0$  because  $I_\varphi(\lambda x\chi_{A_n}) = \infty$  for  $\lambda > b_\varphi/\alpha$  and each  $n$ . We will prove that  $\text{supp } (Ces_\varphi)_a = I$ . Let  $x = \chi_{(a,b)}$  for any  $0 < a < b < m(I)$ . Take a decreasing sequence of Lebesgue measurable sets  $A_n \subset I$  with empty intersection. Then  $\mu(A_n \cap (a, b)) \rightarrow 0$ . Set  $B_n = A_n \cap (a, b)$  and  $x_n = x\chi_{B_n}$ . Then  $Cx_n \rightarrow 0$  uniformly. Consequently,  $\rho_\varphi(\lambda x_n) = I_\varphi(\lambda Cx_n) \rightarrow 0$  for each  $\lambda > 0$ , which for  $I = [0, 1]$  follows directly and for  $I = [0, \infty)$  we need Proposition 3 from [21]. Consequently,  $\|x_n\|_{Ces(\varphi)} \rightarrow 0$ . Thus  $x \in (Ces_\varphi)_a$  which gives  $\text{supp } (Ces_\varphi)_a = I$ .

Finally, suppose  $\limsup_{t \rightarrow 0^+} Cx(t) = \delta > 0$ . Then there is a number  $\gamma > 0$  and a sequence  $(t_k)_{k=1}^\infty$  such that  $t_k \rightarrow 0^+$  and  $Cx(t_k) \geq \delta/2$  for each  $k$ . Since the function  $Cx$  is continuous on the interval  $(0, m(I))$ , for each  $k$  there is a measurable set  $B_k$  of positive measure such that

$$Cx(t) \geq \delta/4$$

for each  $t \in B_k$ . Let  $A_n = (0, \frac{1}{n})$  for  $n \in \mathbb{N}$ . Consequently, for each  $n$  we find a number  $k_n$  with  $\mu(B_{k_n} \cap A_n) > 0$ . Thus

$$\rho_\varphi(\lambda x \chi_{A_n}) = I_\varphi(\lambda C(x \chi_{A_n})) \geq I_\varphi(\lambda(Cx) \chi_{A_n}) \geq I_\varphi(\lambda(Cx) \chi_{B_{k_n} \cap A_n}) = \infty$$

for each  $\lambda > 4b_\varphi/\delta$ , whence  $\|x \chi_{A_n}\|_{Ces(\varphi)} \not\rightarrow 0$ . Thus  $x \notin (Ces_\varphi)_a$ .  $\square$

**Remark 2.** Let  $I = [0, 1]$  or  $I = [0, \infty)$ .

(i) Suppose there is a sequence  $(z_n)_{n=1}^\infty$  in  $Ces_\varphi(I)$  of pairwise disjoint supports satisfying conditions:

(a)  $\|z_n\|_{Ces(\varphi)} = 1$ , for each  $n$ .

(b)  $\|\sup_n z_n\|_{Ces(\varphi)} = 1$ .

Then  $Ces_\varphi(I)$  contains an order isomorphically isometric copy of  $l^\infty$ .

(ii) Assume that  $\varphi < \infty$  and there is an element  $z \in Ces_\varphi(I)$  such that  $\|z\|_{Ces(\varphi)} = 1$  and

$$\delta(z) := \inf \left\{ \lambda > 0 : \rho_\varphi\left(\frac{z}{\lambda}\right) < \infty \right\} = 1.$$

Then  $Ces_\varphi(I)$  contains an order isomorphically isometric copy of  $l^\infty$ .

(iii) Let  $b_\varphi < \infty$ . Suppose there is an element  $z \in Ces_\varphi(I)$  such that  $\|z\|_{Ces(\varphi)} = 1$  and

$$(3.1) \quad \rho_\varphi\left(\frac{z-u}{\lambda}\right) = \infty$$

for all  $\lambda < 1$  and  $u \in (Ces_\varphi)_a$ . Then  $Ces_\varphi(I)$  contains an order isomorphically isometric copy of  $l^\infty$ .

*Proof.* (i) It is enough to apply Theorem 1 in [18]. Note that each Banach ideal space (in particular  $Ces_\varphi$ ) satisfies the assumption of Theorem 1 in [18].

(ii) We apply Theorem 2 from [18]. Note that if  $\varphi < \infty$  then  $(Ces_\varphi)_a = C_\varphi$  (by Theorem 5 from [21]). Next, if  $\varphi < \infty$  and  $Ces_\varphi \neq \{0\}$  then  $\text{supp}(Ces_\varphi)_a = \text{supp } Ces_\varphi = I$  (for  $I = [0, \infty)$  it is enough to apply Proposition 3 from [21]). Moreover, since  $L^\varphi \in (FP)$ , so  $Ces_\varphi \in (FP)$  (see Theorem 1 in [26]), in particular  $Ces_\varphi$  is monotone complete. Consequently, the space  $Ces_\varphi$  satisfies the assumptions of Theorem 2 from [18]. On the other hand,  $Ces_\varphi$  is a modular space generated by a convex modular  $\rho_\varphi$ . By Theorem 2.1 in [17] we get

$$\text{dist}(z, (Ces_\varphi)_a) = \inf \left\{ \lambda > 0 : \rho_\varphi\left(\frac{z}{\lambda}\right) < \infty \right\}.$$

Thus the conclusion follows from Theorem 2 from [18].

(iii) Note that  $\text{supp}(Ces_\varphi)_a = I$  by Lemma 1. Similarly as in (ii) above we conclude that the space  $Ces_\varphi$  satisfies the assumptions of Theorem 2 from [18]. We prove that  $\text{dist}(z, (Ces_\varphi)_a) = 1$ . Clearly,  $\text{dist}(z, (Ces_\varphi)_a) \leq 1$ . We claim that  $\|z - u\|_{Ces(\varphi)} \geq 1$  for each  $u \in (Ces_\varphi)_a$ . Let  $u \in (Ces_\varphi)_a$ . The set  $\{\lambda > 0 : \rho_\varphi(\frac{z-u}{\lambda}) \leq 1\}$  is nonempty. By the assumption (3.1) we have

$$\|z - u\|_{Ces(\varphi)} = \inf \left\{ \lambda > 0 : \rho_\varphi\left(\frac{z-u}{\lambda}\right) \leq 1 \right\} \geq 1,$$

which proves the claim. Thus  $\text{dist}(z, (Ces_\varphi)_a) = 1$  and, by Theorem 2 from [18],  $Ces_\varphi(I)$  contains an order isomorphically isometric copy of  $l^\infty$ .  $\square$

The following characterization of order continuity is well known.

**Theorem A.** (G. Ya. Lozanovskii, see [31]) A Banach ideal space  $E$  is order continuous if and only if  $E$  contains no isomorphic copy of  $l^\infty$ .

**Theorem 3.** Let  $\varphi$  be an Orlicz function.

(i) In the case when  $\varphi(b_\varphi) = \infty$  we assume that for each  $\epsilon > 0$  there exists a constant  $D = D(\epsilon) > 0$  such that

$$(3.2) \quad \rho_\varphi(f) \leq DI_\varphi(f) + \epsilon,$$

for all  $f \in KL^\varphi[0, 1]$  satisfying  $|f(t)| \geq \varphi^{-1}(\epsilon)$  for a.e.  $t \in \text{supp}(f)$  (see (2.1) for the definition). If  $\varphi \notin \Delta_2(\infty)$  then  $Ces_\varphi[0, 1]$  contains an order isomorphically isometric copy of  $l^\infty$ .

(ii) If  $\varphi \in \Delta_2(\infty)$  then  $Ces_\varphi[0, 1]$  does not contain an order isomorphic copy of  $l^\infty$ .

*Proof.* (i). We divide the proof into two parts.

(A) Suppose  $b_\varphi = \infty$ . Since  $\varphi \notin \Delta_2(\infty)$ , for each  $i \in \mathbb{N}$  there is a sequence  $(u_n^i)_{n \in \mathbb{N}} \subset \mathbb{R}_+$  such that  $u_n^i \nearrow \infty$  as  $n \rightarrow \infty$ ,

$$\varphi\left(\left(1 + \frac{1}{i}\right)u_n^i\right) > 2^{n+i}\varphi(u_n^i)$$

and  $\varphi(u_n^i) \geq 1$  for all  $n \in \mathbb{N}$ . Moreover, there exists a sequence of pairwise disjoint intervals  $A^i$  such that  $\bigcup_{i=1}^\infty A^i = [0, 1)$  and  $\mu(A^i) = 2^{-i}$  for  $i \in \mathbb{N}$ . Consequently, we can choose a sequence of intervals  $(A_n^i)_{n=1}^\infty$  with the following properties:

(a)  $(A_n^i) \subset A^i$  for all  $n \in \mathbb{N}$ ,

(b)  $A_n^i \cap A_m^i = \emptyset$  for all  $n \neq m$ ,  $n, m \in \mathbb{N}$ ,

(c)  $\mu(A_n^i) = \frac{2^{-n-i}}{\varphi(u_n^i)}$  for all  $n \in \mathbb{N}$ ,

for all  $i \in \mathbb{N}$ . This is possible, since  $\sum_{n=1}^\infty \frac{2^{-n-i}}{\varphi(u_n^i)} < 2^{-i}$  for all  $i \in \mathbb{N}$ . Finally, for  $i = 1, 2, 3, \dots$  define the elements

$$x_i = \sum_{n=1}^\infty u_n^i \chi_{A_n^i},$$

and  $x = \sum_{i=1}^\infty x_i$ . From the construction it follows that  $\text{supp}(x_i) \cap \text{supp}(x_j) = \emptyset$  for all  $i \neq j$ . Moreover,

$$\begin{aligned} I_\varphi(x_i) &= \int_0^1 \varphi(x_i(t)) dt = \int_0^1 \sum_{n=1}^\infty \varphi(u_n^i) \chi_{A_n^i}(t) dt \\ &= \sum_{n=1}^\infty \varphi(u_n^i) \mu(A_n^i) = \sum_{n=1}^\infty \varphi(u_n^i) \frac{2^{-n-i}}{\varphi(u_n^i)} = 2^{-i}, \end{aligned}$$

and

$$I_\varphi(x) = \int_0^1 \sum_{i=1}^\infty \varphi(x_i(t)) dt = \sum_{i=1}^\infty \int_0^1 \varphi(x_i(t)) dt = 1.$$

Without loss of generality we may assume that  $x_i$  is increasing for each  $i \in \mathbb{N}$ . Consider elements  $u_n^i$  in the following way

$$(3.3) \quad \begin{array}{cccccc} u_1^1 & u_2^1 & u_3^1 & \cdots & u_n^1 & \cdots \\ u_1^2 & u_2^2 & u_3^2 & \cdots & u_n^2 & \cdots \\ u_1^3 & u_2^3 & u_3^3 & \cdots & u_n^3 & \cdots \\ \vdots & \vdots & \vdots & \ddots & \vdots & \ddots \end{array}$$

Order the set  $\{u_1^1, u_2^1, u_3^1\}$  from the smallest to the largest in the sense of order  $\leq$ . Denote ordered elements by  $v_1, v_2$  and  $v_3$ . That is,  $v_1 \leq v_2 \leq v_3$ . Since for each  $i \in \mathbb{N}$ ,  $u_n^i \rightarrow \infty$  as  $n \rightarrow \infty$ , so there exist indices  $i_1^2, i_2^2, i_3^2 \in \mathbb{N}$ ,  $i_1^2 > 2$  and  $i_2^2 > 1$  such that  $u_{i_1^2}^1, u_{i_2^2}^2, u_{i_3^2}^3 \geq v_3$ . Order the set  $\{u_{i_1^2}^1, u_{i_2^2}^2, u_{i_3^2}^3\}$  as before and denote ordered elements by  $v_4, v_5$  and  $v_6$ . Clearly, the sequence  $(v_n)_{n=1}^6$  is non-decreasing. Generally, in the  $m$ -th step we find indexes  $i_1^m, i_2^m, \dots, i_{m+1}^m \in \mathbb{N}$ ,  $i_1^m \geq i_1^{m-1}$ ,  $i_2^m \geq i_2^{m-1}, \dots, i_m^m \geq i_m^{m-1}$  such that  $u_{i_1^m}^1, u_{i_2^m}^2, \dots, u_{i_{m+1}^m}^{m+1} \geq v_{l_m}$  and, ordering the set

$\{u_1^1, u_2^2, \dots, u_{m+1}^{m+1}\}$ , we obtain a sequence  $(v_i)_{i=1}^{l_m}$ , where  $l_1 = 3$  and  $l_m = l_{m-1} + m + 1$  for  $m \geq 2$ , which is non-decreasing. By the induction, we construct a sequence  $(v_n)_{n=1}^\infty$  which is non-decreasing. Corresponding to each value  $v_n$  the respective interval  $A_n^i$  we can denote by  $A_n$  for simplicity. Define an element

$$y = \sum_{n=1}^{\infty} y_n,$$

where  $y_n = v_n \chi_{B_n}$ ,  $B_n = \left(a - \sum_{m=1}^n \mu(A_m), a - \sum_{m=1}^{n-1} \mu(A_m)\right)$  and  $a = \sum_{n=1}^{\infty} \mu(A_n)$ . We have  $I_\varphi(y) \leq I_\varphi(x) \leq 1$ . Applying condition (3.2) for  $\epsilon = 1/2$  we find  $t \in (0, 1]$  such that  $I_\varphi(z) \leq 1/2D$  and  $|z(t)| \geq \varphi^{-1}(1/2)$  for a.e.  $t \in \text{supp}(z)$ , where  $z = y \chi_{[0,t]}$ . Therefore, we have

$$\rho_\varphi(z) \leq DI_\varphi(z) + 1/2 = 1.$$

Moreover, there exists  $n_0 \in \mathbb{N}$  such that  $\sum_{n=n_0}^{\infty} \mu(B_n) \leq t$ . Set  $\lambda > 0$ . There is  $i_0$  with  $\lambda \geq 1/i_0$ . We have

$$\rho_\varphi((1+\lambda)z) = I_\varphi(C((1+\lambda)z)) \geq I_\varphi((1+\lambda)z) \geq I_\varphi\left(\left(1 + \frac{1}{i_0}\right)z\right).$$

Consider the modular  $I_\varphi((1+1/i_0)z)$ . By the construction of element  $z$ , we have chosen infinitely many terms from the  $i_0$ -th row of matrix (3.3). Denote them by  $u_{n_k}^{i_0}$  for  $k = 1, 2, \dots$ . We have

$$\varphi\left(\left(1 + \frac{1}{i_0}\right)u_{n_k}^{i_0}\right) > 2^{n_k+i_0}\varphi(u_{n_k}^{i_0}),$$

for each  $k$  and  $\mu(A_{n_k}^{i_0}) = 2^{-n_k-i_0}/\varphi(u_{n_k}^{i_0})$ . Therefore,

$$I_\varphi\left(\left(1 + \frac{1}{i_0}\right)z\right) \geq \sum_{k=k_0}^{\infty} \varphi\left(\left(1 + \frac{1}{i_0}\right)u_{n_k}^{i_0}\right) \mu(A_{n_k}^{i_0}) \geq \sum_{k=k_0}^{\infty} 1 = \infty.$$

Consequently,  $\|z\|_{Ces(\varphi)} = 1$  and, by Remark 2(ii), we conclude that  $Ces_\varphi[0, 1]$  contains an order isomorphically isometric copy of  $l^\infty$ .

(B) Suppose  $b_\varphi < \infty$ . We apply Remark 2(i).

(B1) Let  $\varphi(b_\varphi) = \infty$ . Let  $(u_n) \subset \mathbb{R}_+$  be the sequence with  $u_n \nearrow b_\varphi^-$ . Since  $\varphi(u_n) \rightarrow \infty$ , therefore we can assume that  $\varphi(u_n) \geq 1$  for all  $n \in \mathbb{N}$ . Put  $a_n = 1/2^n \varphi(u_n)$  for  $n \in \mathbb{N}$  and denote  $a = \sum_{n=1}^{\infty} a_n$ . Define a sequence of pairwise disjoint open intervals  $(A_n)_{n \in \mathbb{N}} \subset [0, 1]$  as follows

$$A_n = \left(a - \sum_{k=1}^n \frac{1}{2^k \varphi(u_k)}, a - \sum_{k=1}^{n-1} \frac{1}{2^k \varphi(u_k)}\right),$$

for all  $n \in \mathbb{N}$ . Let

$$x = \sum_{n=1}^{\infty} u_n \chi_{A_n}.$$

We have

$$\begin{aligned} I_\varphi(x) &= \int_0^\infty \varphi(x) d\mu = \int_0^\infty \sum_{n=1}^{\infty} \varphi(u_n) \chi_{A_n} d\mu \\ &= \sum_{n=1}^{\infty} \int_{A_n} \varphi(u_n) d\mu = \sum_{n=1}^{\infty} \varphi(u_n) \mu(A_n) = \sum_{n=1}^{\infty} \frac{1}{2^n} = 1. \end{aligned}$$

Now, like in (A), we can find  $t \in (0, 1]$  with  $\rho_\varphi(z) \leq 1$ , where  $z = x\chi_{[0,t]}$ . Hence, relabeling if necessary, we may assume that

$$z = \sum_{n=1}^{\infty} u_n \chi_{A_n}.$$

Because for all  $\lambda > 0$  there exists  $N = N(\lambda) \in \mathbb{N}$  such that for all  $k \geq N$ ,  $(1+\lambda)u_k > b_\varphi$ , hence

$$\begin{aligned} \rho_\varphi((1+\lambda)z) &= I_\varphi(C((1+\lambda)z)) = I_\varphi((1+\lambda)Cz) \\ &\geq I_\varphi((1+\lambda)z) = \sum_{n=1}^{\infty} \varphi((1+\lambda)u_n)\mu(A_n) = \infty. \end{aligned}$$

Therefore  $\|z\|_{Ces(\varphi)} = 1$ . We will show that

$$\rho_\varphi(\lambda(z-u)) = \infty$$

for all  $\lambda > 1$  and  $u \in (Ces_\varphi)_a$  which would imply that  $Ces_\varphi[0, 1]$  contains an order isomorphically isometric copy of  $l^\infty$  (see Remark 2(iii)). Let  $\lambda > 1$  and  $u \in (Ces_\varphi)_a$ . We have

$$\rho_\varphi(\lambda(z-u)) = I_\varphi(C(|\lambda(z-u)|)) \geq I_\varphi(\lambda C|z| - |u|).$$

By Lemma 1,  $\lim_{t \rightarrow 0^+} C|u|(t) = 0$ . Clearly,  $Cz \geq z$ , because  $z$  is nonincreasing. Then, since the function  $C|u|$  is continuous, there is  $\gamma = \gamma(\lambda) > 0$  such that

$$\lambda Cz(t) \geq \lambda z(t) > b_\varphi + \lambda C|u|(t)$$

for  $t \in (0, \gamma)$ . Consequently,

$$I_\varphi(\lambda C(|z| - |u|)) \geq \int_0^\gamma \varphi(\lambda(Cz(t) - C|u|(t))) dt = \infty,$$

which finishes the proof.

(B2) Suppose  $\varphi(b_\varphi) < \infty$ . Let us define an element  $x = b_\varphi \chi_{[0,1]}$ . Then  $x \in Ces_\varphi[0, 1]$ . Indeed,

$$\rho_\varphi(x) = I_\varphi(Cx) = I_\varphi(x) = \int_0^1 \varphi(b_\varphi) d\mu = \varphi(b_\varphi) < \infty.$$

There exists  $t > 0$  with

$$\rho_\varphi(x\chi_{[0,t]}) = \int_0^t \varphi(b_\varphi) ds + \int_t^1 \varphi\left(t \frac{b_\varphi}{s}\right) ds \leq 1.$$

Moreover, for each  $\lambda > 0$

$$\rho_\varphi((1+\lambda)x\chi_{[0,t]}) \geq t\varphi((1+\lambda)b_\varphi) = \infty.$$

For  $z = x\chi_{[0,t]}$  we have  $\|z\|_{Ces(\varphi)} = 1$ . Then, like in (B1) above, we conclude that  $Ces_\varphi[0, 1]$  contains an order isomorphically isometric copy of  $l^\infty$ .

(ii). If  $\varphi \in \Delta_2(\infty)$  then  $L^\varphi[0, 1] \in (OC)$  (see for example [33, p. 21]). Consequently,  $Ces_\varphi[0, 1] \in (OC)$  (see [28, Lemma 1 (a)]). By Theorem A,  $Ces_\varphi[0, 1]$  cannot contain isomorphic copy of  $l^\infty$ .  $\square$

**Theorem 4.** Let  $\varphi$  be an Orlicz function.

(i) In the case when  $\varphi(b_\varphi) = \infty$  we assume that there exists constant  $D > 0$  such that

$$(3.4) \quad \rho_\varphi(f) \leq DI_\varphi(f),$$

for all  $f \in KL^\varphi[0, \infty)$  (see (2.1)). If  $\varphi \notin \Delta_2(\mathbb{R}_+)$  then the space  $Ces_\varphi[0, \infty)$  contains an order isomorphically isometric copy of  $l^\infty$ .

(ii) If  $\varphi \in \Delta_2(\mathbb{R}_+)$  then  $Ces_\varphi[0, 1]$  does not contain an order isomorphic copy of  $l^\infty$ .

*Proof.* (i). We divide the proof into three parts.

(A) Assume that  $\varphi > 0$  and  $\varphi < \infty$ . Suppose  $\varphi \notin \Delta_2(\mathbb{R}_+)$ . This means that  $\varphi \notin \Delta_2(\infty)$  or  $\varphi \notin \Delta_2(0)$ . In the first case we can proceed as in the proof of Theorem 3. The proof in the second case is similar to the proof of Theorem 1 in [19]. We present the details for the reader's convenience.

It is well known that  $\varphi \in \Delta_2(0)$  if and only if there exist  $L > 1$  and  $K, u_0 > 0$  such that  $\varphi(u_0) > 0$ , and  $\varphi(Lu) \leq K\varphi(u)$  for all  $u \in [0, u_0]$  (see [8, p. 9]). Assume that  $\varphi \notin \Delta_2(0)$ . For each  $L > 1$  and every sequence  $(K_n)_{n=1}^\infty$  we find a sequence  $u_n \searrow 0^+$  with

$$\varphi(Lu_n) > K_n \varphi(u_n).$$

Take a sequence  $(K_n)_{n=1}^\infty$  satisfying

$$\sum_{n=1}^\infty \frac{1}{K_n} < \infty.$$

Let  $(\epsilon_m)_{m=1}^\infty$  be any positive, decreasing sequence converging to zero. For any  $m \in \mathbb{N}$  take a sequence  $(K_n^m)_{n=1}^\infty$  of positive reals numbers such that

$$\sum_{n=1}^\infty \frac{1}{K_n^m} \leq \frac{1}{2^m}.$$

Now by the first part, for any  $m \in \mathbb{N}$  we can find a decreasing sequence  $(u_n^m)_{n=1}^\infty$  with  $u_n^m \rightarrow 0$  as  $n \rightarrow \infty$  and  $\varphi((1 + \epsilon_m)u_n^m) > K_n^m \varphi(u_n^m)$  for all  $m \in \mathbb{N}$ . Thus

$$\sum_{n=1}^\infty \frac{\varphi(u_n^m)}{\varphi((1 + \epsilon_m)u_n^m)} < \sum_{n=1}^\infty \frac{1}{K_n^m} \leq \frac{1}{2^m},$$

for all  $m \in \mathbb{N}$ . In view of  $u_n^m \rightarrow 0$  as  $n \rightarrow \infty$ , we can find a subsequence  $(n_k) \subset \mathbb{N}$  such that

$$u_{n_1}^1 > u_{n_2}^2 > u_{n_3}^3 > \dots$$

Hence, without loss of generality, we can assume that for  $m > 1$ ,

$$u_m^m < u_{m-1}^{m-1}.$$

Let

$$c_n = \frac{1}{\varphi((1 + \epsilon_n)u_n^n)},$$

for  $n \in \mathbb{N}$ . Note that  $1 \leq c_n < \infty$  for all  $n \in \mathbb{N}$ . Define

$$f(t) = \sum_{n=1}^\infty u_n^n \chi_{\Omega_n}(t),$$

where  $\Omega_n = [c_1 + c_2 + \dots + c_{n-1}, c_1 + c_2 + \dots + c_{n-1} + c_n) \subset \mathbb{R}_+$ . It is clear that the function  $f$  is decreasing. We have

$$\begin{aligned} I_\varphi(f) &= \int_0^\infty \varphi(|f(t)|) dt = \sum_{n=1}^\infty c_n \varphi(u_n^n) \\ &\leq \sum_{n=1}^\infty \frac{\varphi(u_n^n)}{\varphi((1 + \epsilon_n)u_n^n)} \leq \sum_{n=1}^\infty \frac{1}{2^n} = 1. \end{aligned}$$

For any  $\epsilon > 0$  and for any positive integer  $M$ , we get

$$\begin{aligned} \int_M^\infty \varphi((1 + \epsilon)|f(t)|) dt &\geq \int_{M_1}^\infty \varphi((1 + \epsilon)|f(t)|) dt \\ &= \sum_{n=n_1}^\infty c_n \varphi((1 + \epsilon)u_n^n) = \sum_{n=n_1}^\infty \frac{\varphi((1 + \epsilon)u_n^n)}{\varphi((1 + \epsilon_n)u_n^n)} \geq \sum_{n=n_2}^\infty 1 = \infty, \end{aligned}$$



for some  $M_1 \geq M$  and  $n_2 \geq n_1$ . Since  $I_\varphi(f) \leq 1$ , there is  $N \in (0, \infty)$  such that

$$I_\varphi(f\chi_{[N, \infty)}) \leq \frac{1}{D}.$$

Consequently, by the assumption,  $\rho_\varphi(f\chi_{[N, \infty)}) \leq 1$ . Put  $g = f\chi_{[N, \infty)}$ . Since  $f$  is a decreasing function, for any  $\epsilon > 0$ ,

$$\begin{aligned} (1 + \epsilon) \frac{1}{t} \int_N^t |f(x)| dx &\geq (1 + \epsilon) \left( \frac{t - N}{t} \right) |f(t)| \\ &= (1 + \epsilon) \left( 1 - \frac{N}{t} \right) |f(t)| \geq \left( 1 + \frac{\epsilon}{2} \right) |f(t)|, \end{aligned}$$

for  $t$  large enough. Now, for any  $\epsilon > 0$  and  $N_1$  large enough we get

$$\begin{aligned} \rho_\varphi((1 + \epsilon)g) &= \int_N^\infty \varphi \left( (1 + \epsilon) \frac{1}{t} \int_N^t |f(x)| dx \right) dt \\ &\geq \int_{N_1}^\infty \varphi \left( \left( 1 + \frac{\epsilon}{2} \right) |f(t)| \right) dt = \infty. \end{aligned}$$

We have constructed an element  $g \in Ces_\varphi[0, \infty)$  such that  $\rho_\varphi(g) \leq 1$  and for every  $\epsilon > 0$ ,  $\rho_\varphi((1 + \epsilon)g) = \infty$ . By Remark 2 we conclude that  $Ces_\varphi[0, \infty)$  contains an order isomorphically isometric copy of  $l^\infty$ .

(B) Suppose  $b_\varphi < \infty$  and

(B1)  $\varphi(b_\varphi) = \infty$ . The proof is similar to the proof of case (B1) of Theorem 3.

(B2)  $\varphi(b_\varphi) < \infty$ . We just proceed as in the proof of Theorem 3 case (B2). We additionally use the assumption  $Ces_\varphi \neq \{0\}$  and Proposition 3 from [21].

(C) Assume that  $\varphi < \infty$  and  $a_\varphi > 0$ . Put  $x = a_\varphi \chi_{[0, \infty)}$ . Then  $\|x\|_{Ces(\varphi)} = 1$  and

$$\delta(x) = \inf\{\lambda > 0 : \rho_\varphi(x/\lambda) < \infty\} = 1.$$

Indeed, for any  $\lambda > 0$

$$\rho_\varphi((1 + \lambda)x) = I_\varphi((1 + \lambda)Cx) = I_\varphi((1 + \lambda)x) = \varphi((1 + \lambda)a_\varphi)m([0, \infty)) = \infty.$$

Therefore, by Remark 2(ii),  $Ces_\varphi[0, \infty)$  contains an order isomorphically isometric copy of  $l^\infty$ . (ii). We can use analogue arguments as in the proof of Theorem 3.  $\square$

Now we will discuss the technical assumption from Theorem 3 and Theorem 4.

Let us recall the standard form of the integral Hardy inequality (see [23]). Suppose  $p > 1$  and  $f$  is a nonnegative  $p$ -integrable function on  $I$ . Then  $f$  is integrable over the interval  $(0, x) \cap I$  for each positive  $x$  and

$$\int_I \left( \frac{1}{x} \int_0^x f(t) dt \right)^p dx \leq \left( \frac{p}{p-1} \right)^p \int_I f(x)^p dx.$$

The dilation operator  $D_s$ ,  $s > 0$ , defined on  $L^0(I)$  by

$$D_s x(t) = x(t/s) \chi_I(t/s) = x(t/s) \chi_{[0, \min\{1, s\})}(t),$$

for  $t \in I$ , is bounded in any symmetric space  $E$  on  $I$  and  $\|D_s\|_{E \rightarrow E} \leq \max\{1, s\}$  (see [5, p. 148]). For the theory of symmetric (rearrangement invariant) spaces the reader is referred to [5] and [25]. Moreover, the lower and upper Boyd indices of  $E$  are defined by

$$\begin{aligned} p(E) &= \lim_{s \rightarrow 0^+} \frac{\ln \|D_s\|_{E \rightarrow E}}{\ln s}, \\ q(E) &= \lim_{s \rightarrow \infty} \frac{\ln \|D_s\|_{E \rightarrow E}}{\ln s}. \end{aligned}$$

In particular, they satisfy the inequalities

$$1 \leq p(E) \leq q(E) \leq \infty.$$

In the case when  $E$  is the Orlicz space  $L^\varphi$ , these indices equal to the so-called lower and upper Orlicz-Matuszewska indices  $\alpha_\varphi$  and  $\beta_\varphi$  of Orlicz functions generating the Orlicz spaces, i.e.,  $\alpha_\varphi = p(L^\varphi)$  and  $\beta_\varphi = q(L^\varphi)$  (see [30, Proposition 2.b.5 and Remark 2 on page 140]). For more details see [6], [7], [32], [33] and [36].

Let us mention the important result about boundedness of the operator  $C$ .

**Theorem B.** [23, p. 127] For any symmetric space  $E$  on  $I$  the operator  $C : E \rightarrow E$  is bounded if and only if the lower Boyd index satisfies  $p(E) > 1$ .

Note that if  $(X, \|\cdot\|_X)$  is a Banach ideal space then  $C : X \rightarrow X$  implies  $C$  is bounded. In fact, if  $C : X \rightarrow X$ , then  $X \hookrightarrow CX$ . This means that there is  $M > 0$  with  $\|x\|_{CX} = \|C|x|\|_X \leq M \|x\|_X$  for all  $x \in X$ , i.e.  $C$  is bounded. However, it may happen that  $X \not\hookrightarrow CX$  (see [16, Proposition 2.1]). Moreover,  $C : CX \rightarrow X$  is always bounded (from the definition of  $CX$ ) and  $CX$  is so-called optimal domain of  $C$  for  $X$  (cf. [16] and [28]). The immediate consequence of Theorem B and the above discussion about indices of the Orlicz space  $L^\varphi$  is a next corollary.

**Corollary 5.** The embedding  $L^\varphi \hookrightarrow Ces_\varphi$  holds if and only if  $\alpha_\varphi > 1$ .

**Remark 6.** If  $f \in KL^\varphi[0, 1]$  then

$$\int_0^t f(s)ds < \infty$$

for each  $t \in [0, 1]$ , because  $KL^\varphi[0, 1] \hookrightarrow L^\varphi[0, 1] \hookrightarrow L^1[0, 1]$ . However, we present also the direct proof, since we need this fact also in the proof of Proposition 8 in the case of  $I = [0, \infty)$ . Conversely, suppose  $\int_0^{t_0} |f(s)|ds = \infty$  for some  $t_0 \in (0, 1]$ . Consequently,  $f$  is unbounded on  $(0, t_0)$ . Since  $\varphi(u)/u \nearrow a$ ,  $a \leq \infty$ , then there is  $a_0 > 0$  such that  $\varphi(u) \geq a_0 u/2$  for  $u \geq u_3$ . Setting  $I_0 = \{t \in [0, 1] : |f(t)| \geq u_3\}$  we have

$$I_\varphi(f) \geq \int_{I_0} \varphi(|f(s)|)ds \geq \frac{a_0}{2} \int_{I_0 \cap [0, t_0]} |f(s)|ds = \infty,$$

whence  $f \notin KL^\varphi[0, 1]$  and the claim is proved.

**Proposition 7.** Let  $\varphi$  be an Orlicz function with  $\varphi < \infty$ . Consider the following conditions:

- (i) There exists  $p > 1$ , a convex function  $\gamma$  and constants  $A, B, u_0 > 0$  such that  $\varphi(u_0) > 0$  and for all  $u \in [u_0, \infty)$ ,

$$A\gamma(u) \leq \varphi(u)^{1/p} \leq B\gamma(u),$$

i.e.  $\varphi^{1/p}$  is equivalent to a convex function for "large arguments".

- (ii) For each  $\epsilon > 0$  there exists a constant  $D = D(\epsilon) > 0$  such that

$$\rho_\varphi(f) \leq DI_\varphi(f) + \epsilon,$$

for all  $f \in KL^\varphi[0, 1]$  satisfying  $|f(t)| \geq \varphi^{-1}(\epsilon)$  for a.e.  $t \in \text{supp}(f)$ . Moreover, the Orlicz class  $KL^\varphi[0, 1]$  is closed under the operator  $C$ .

- (iii)  $C : L^\varphi[0, 1] \rightarrow L^\varphi[0, 1]$ .

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii).

*Proof.* (i) $\Rightarrow$ (ii). Take  $\epsilon > 0$ . Let  $a_\varphi < u_1$  be such that  $\varphi(u_1) = \epsilon$ .

Suppose  $u_1 < u_0$  ( $u_0$  is from the condition (i)). We claim that we can extend the domain of the function  $\gamma$  to the interval  $[u_1, \infty)$  such that the extended function  $\gamma_e : [u_1, \infty) \rightarrow (0, \infty)$  is still continuous and convex. Denote by  $\gamma'_+$  the right derivative of  $\gamma$ . Since  $\lim_{u \rightarrow \infty} \varphi(u)^{1/p} = \infty$ , the condition (i) implies that there is  $u_2 \geq u_0$  such that  $\gamma'_+(u_2) \geq 0$  and  $\gamma(u_2) = \gamma(u_0)$ . Taking  $\gamma_e := \gamma(u_0)\chi_{[u_1, u_2]} + \gamma\chi_{(u_2, \infty)}$  proves the claim.

Moreover, there exist new constants  $A', B' > 0$  such that

$$A'\gamma_e(u) \leq \varphi(u)^{1/p} \leq B'\gamma_e(u),$$

for all  $u \geq u_1$ . In fact, it is sufficient to take  $A' = \min\{A, A''\}$  and  $B' = \max\{B, B''\}$ , where

$$A'' = \min\{\varphi(u)^{1/p}/\gamma_e(u) : u_1 \leq u \leq u_2\} \text{ and } B'' = \max\{\varphi(u)^{1/p}/\gamma_e(u) : u_1 \leq u \leq u_2\}$$

(these numbers exist because  $\varphi(u)^{1/p}/\gamma_e(u)$  is continuous function on closed interval  $[u_1, u_2]$ ).

If  $u_1 \geq u_0$  then we proceed with  $\gamma_e = \gamma$ ,  $A' = A$  and  $B' = B$ .

Let  $f \in KL^\varphi[0, 1]$  be such that  $|f(t)| \geq u_1$ , for a.e.  $t \in \text{supp}(f)$ . Set

$$A_0 = \{t \in [0, 1] : C|f|(t) \geq u_1\}$$

and

$$B_0 = [0, 1] \setminus A_0.$$

Thus, by Remark 6,

$$C|f|(t) = \frac{1}{t} \int_0^t |f(s)| ds = \int_0^1 |f(tu)| du < \infty,$$

for each  $t > 0$ . Then, by (i),

$$\begin{aligned} \rho_\varphi(f) &= I_\varphi(C|f|) \\ &= \int_0^1 \varphi \left( \int_0^1 |f(tu)| du \right) dt = \int_{A_0} \varphi \left( \int_0^1 |f(tu)| du \right) dt + \int_{B_0} \varphi \left( \int_0^1 |f(tu)| du \right) dt \\ &\leq (B')^p \int_{A_0} \left( \gamma_e \left( \int_0^1 |f(tu)| du \right) \right)^p dt + \varphi(u_1) \\ &\leq (B')^p \int_{A_0} \left( \gamma_e \left( \int_0^1 |f(tu)| du \right) \right)^p dt + \epsilon. \end{aligned}$$

For each  $t \in [0, 1]$ , let

$$C_0^t = \{u \in [0, 1] : |f(tu)| \geq u_1\} \text{ and } D_0^t = [0, 1] \setminus C_0^t.$$

Since  $|f(s)| \geq u_1$ , for a.e.  $s \in \text{supp } f$ , so

$$(3.5) \quad u_1 \leq C|f|(t) = \int_0^1 |f(tu)| du = \int_{C_0^t} |f(tu)| du$$

for  $t \in A_0$ . Consequently, by Hardy and Jensen inequality, we get

$$\begin{aligned} \rho_\varphi(f) &= I_\varphi(C|f|) \leq (B')^p \int_{A_0} \left( \int_{C_0^t} \gamma_e(|f(tu)|) du \right)^p dt + \epsilon \\ &\leq (B')^p \int_{A_0} \left( \int_{C_0^t} (A')^{-1} \varphi(|f(tu)|)^{1/p} du \right)^p dt + \epsilon \\ &\leq (B')^p \int_0^1 \left( \int_{C_0^t} (A')^{-1} \varphi(|f(tu)|)^{1/p} du \right)^p dt + \epsilon \\ &\leq \left( \frac{B'}{A'} \right)^p \int_0^1 \left( \int_0^t \varphi(|f(s)|)^{1/p} \frac{ds}{t} \right)^p dt + \epsilon \end{aligned}$$

$$\begin{aligned}
&= \left( \frac{B'}{A'} \right)^p \int_0^1 \left( C(\varphi(|f|)^{1/p})(t) \right)^p dt + \epsilon \leq \left( \frac{B'p}{A'(p-1)} \right)^p \int_0^1 \varphi(|f(t)|) dt + \epsilon \\
&= \left( \frac{B'p}{A'(p-1)} \right)^p I_\varphi(f) + \epsilon.
\end{aligned}$$

The proof is finished with  $D = \left( \frac{B'p}{A'(p-1)} \right)^p$  and the constant  $D$  depends only on  $\epsilon$ .

Finally, we will show that the condition (i) implies that the Orlicz class  $KL^\varphi[0, 1]$  is closed under the operator  $C$ . Take  $f \in KL^\varphi[0, 1]$ . We apply the above proof with the function  $\gamma$ . Denote

$$I_1 = \{t \in [0, 1] : |f(t)| \geq u_0\}, I_2 = \{t \in [0, 1] : |f(t)| < u_0\}$$

and

$$\tilde{f} = |f| \chi_{I_1} + u_0 \chi_{I_2}.$$

Clearly,  $|f| \leq \tilde{f}$ . Moreover,

$$\rho_\varphi(f) = I_\varphi(C|f|) \leq I_\varphi(C\tilde{f}) = \int_0^1 \varphi \left( \int_0^1 \tilde{f}(tu) du \right) dt.$$

Note that  $\tilde{f} \geq u_0$  whence  $\int_0^1 \tilde{f}(tu) du \geq u_0$  for  $t \in [0, 1]$ . Consequently, by Hardy and Jensen inequality, similarly as in the above proof, applying condition (i) we obtain

$$\begin{aligned}
\rho_\varphi(f) &\leq B^p \int_0^1 \left( \gamma \left( \int_0^1 \tilde{f}(tu) du \right) \right)^p dt \leq B^p \int_0^1 \left( \int_0^1 \gamma(\tilde{f}(tu)) du \right)^p dt \\
&\leq \left( \frac{Bp}{A(p-1)} \right)^p I_\varphi(\tilde{f}) = \left( \frac{Bp}{A(p-1)} \right)^p \int_0^1 \varphi(|f| \chi_{I_1} + u_0 \chi_{I_2}) \\
&= \left( \frac{Bp}{A(p-1)} \right)^p (I_\varphi(|f| \chi_{I_1}) + \varphi(u_0)) \leq \left( \frac{Bp}{A(p-1)} \right)^p (I_\varphi(f) + \varphi(u_0)) < \infty.
\end{aligned}$$

(ii) $\Rightarrow$ (iii). Take  $x \in L^\varphi[0, 1]$ . Of course,  $I_\varphi(\lambda x) < \infty$  for some  $\lambda > 0$ . This means that  $\lambda x \in KL^\varphi[0, 1]$ . Therefore  $\rho_\varphi(\lambda x) < \infty$  from (ii). Consequently,  $I_\varphi(C|\lambda x|) < \infty$ , i.e.  $C(\lambda x) \in L^\varphi[0, 1]$ . Because operator  $C$  is homogeneous we have  $\lambda Cx \in L^\varphi[0, 1]$ . Whence  $Cx \in L^\varphi[0, 1]$ .  $\square$

**Proposition 8.** Let  $\varphi$  be an Orlicz function with  $\varphi < \infty$ . Consider the following conditions:

- (i) There exists  $p > 1$ , a convex function  $\gamma$  and constants  $A, B > 0$  such that for all  $u \in [0, \infty)$

$$A\gamma(u) \leq \varphi(u)^{1/p} \leq B\gamma(u),$$

i.e.  $\varphi^{1/p}$  is equivalent to a convex function for "all arguments".

- (ii) There exists a constant  $D > 0$  such that

$$\rho_\varphi(f) \leq DI_\varphi(f),$$

for all  $f \in KL^\varphi[0, \infty)$ . In particular, the Orlicz class  $KL^\varphi[0, \infty)$  is closed under the operator  $C$ .

- (iii)  $C : L^\varphi[0, \infty) \rightarrow L^\varphi[0, \infty)$ .  
(iv) There exists  $x_0 \in [0, \infty)$  such that  $\int_{x_0}^\infty \varphi(t^{-1}) dt < \infty$ .

We have the following implications: (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv).

*Proof.* The proof of implication (i) $\Rightarrow$ (ii) follows by Jensen inequality and Hardy inequality and is similar to the proof of the same implication in Proposition 7.

(ii) $\Rightarrow$ (iii). We go analogously as in the respective proof of Proposition 7.

(iii) $\Rightarrow$ (iv). There exists  $x_0$  such that  $\varphi(x_0^{-1}) < \infty$ . If  $f = x_0^{-1}\chi_{[0,x_0]}$ , then  $I_\varphi(f) < \infty$ , i.e.  $f \in L^\varphi[0,1]$  and

$$Cf(t) = \frac{1}{t} \int_0^t x_0^{-1}\chi_{[0,x_0]}(s)ds = x_0^{-1}\chi_{[0,x_0]}(t) + \frac{1}{t}\chi_{[x_0,\infty)}(t).$$

Therefore, by (iii),  $Cf \in L^\varphi[0,1]$  and

$$\rho_\varphi(\lambda f) = I_\varphi(\lambda Cf) = x_0\varphi(\lambda x_0^{-1}) + \int_{x_0}^\infty \varphi\left(\frac{\lambda}{t}\right) dt < \infty,$$

for some  $\lambda > 0$ . After changing variables in the last integral we get

$$\lambda \int_{x_0/\lambda}^\infty \varphi\left(\frac{1}{u}\right) du < \infty,$$

and the proof is finished.  $\square$

Note that if  $b_\varphi < \infty$  and  $\varphi(b_\varphi) = \infty$  we also apply conditions (3.2) and (3.4) to prove the existence of isometric copy of  $l^\infty$  in  $Ces_\varphi$ . Thus we should discuss similar conditions from Proposition 7 and 8 in that case. Below we add also the case  $b_\varphi < \infty$ ,  $\varphi(b_\varphi) < \infty$ .

**Proposition 9.** Let  $I = [0, \infty)$  and  $\varphi$  be an Orlicz function. Consider the following conditions:

- (i) Let  $b_\varphi < \infty$ ,  $\varphi(b_\varphi) < \infty$  and there exists  $p > 1$ , a convex function  $\gamma$  and constants  $A, B > 0$  such that for all  $u \in [0, b_\varphi]$

$$A\gamma(u) \leq \varphi(u)^{1/p} \leq B\gamma(u).$$

- (ii) Let  $b_\varphi < \infty$ ,  $\varphi(b_\varphi) = \infty$  and there exists  $p > 1$ , a convex function  $\gamma$  and constants  $A, B > 0$  such that for all  $u \in [0, b_\varphi]$

$$A\gamma(u) \leq \varphi(u)^{1/p} \leq B\gamma(u).$$

- (iii) There exists a constant  $D > 0$  such that

$$\rho_\varphi(f) \leq DI_\varphi(f),$$

for all  $f \in KL^\varphi(I)$ . In particular, the Orlicz class  $KL^\varphi(I)$  is closed under the operator  $C$ .

- (iv)  $C : L^\varphi(I) \rightarrow L^\varphi(I)$ .

- (v) there exists  $x_0 \in [0, \infty)$  such that  $\int_{x_0}^\infty \varphi(t^{-1})dt < \infty$ .

We have the following implications: (i)  $\Rightarrow$  (iii)  $\Rightarrow$  (iv)  $\Rightarrow$  (v) and (ii)  $\Rightarrow$  (iii).

*Proof.* It is enough to observe that if  $f \in KL^\varphi$  then  $f[\text{supp}(f)] \subset [0, \varphi(b_\varphi)]$  if  $\varphi(b_\varphi) < \infty$  and  $f[\text{supp}(f)] \subset [0, \varphi(b_\varphi))$  if  $\varphi(b_\varphi) = \infty$  (up to the set of measure zero). Now we apply the proofs of Propositions 7 and 8.  $\square$

**Proposition 10.** Let  $I = [0, 1]$  and  $\varphi$  be an Orlicz function. Consider the following conditions:

- (i) Let  $b_\varphi < \infty$ ,  $\varphi(b_\varphi) < \infty$  and there exists  $p > 1$ , a convex function  $\gamma$  and constants  $A, B, \varphi(u_0) > 0$  such that for all  $u \in [u_0, b_\varphi]$

$$A\gamma(u) \leq \varphi(u)^{1/p} \leq B\gamma(u).$$

- (ii) Let  $b_\varphi < \infty$ ,  $\varphi(b_\varphi) = \infty$  and there exists  $p > 1$ , a convex function  $\gamma$  and constants  $A, B, \varphi(u_0) > 0$  such that for all  $u \in [u_0, b_\varphi]$

$$A\gamma(u) \leq \varphi(u)^{1/p} \leq B\gamma(u).$$

(iii) For each  $\epsilon \in (0, \varphi(b_\varphi))$  there exists a constant  $D > 0$  such that

$$\rho_\varphi(f) \leq DI_\varphi(f) + \epsilon,$$

for all  $f \in KL^\varphi(I)$  satisfying  $|f(t)| \geq \varphi^{-1}(\epsilon)$  for a.e.  $t \in \text{supp}(f)$ . Moreover, the Orlicz class  $KL^\varphi[0, 1]$  is closed under the operator  $C$ .

(iv) For each  $\epsilon > 0$  there exists a constant  $D > 0$  such that

$$\rho_\varphi(f) \leq DI_\varphi(f) + \epsilon,$$

for all  $f \in KL^\varphi(I)$  satisfying  $|f(t)| \geq \varphi^{-1}(\epsilon)$  for a.e.  $t \in \text{supp}(f)$ . Moreover, the Orlicz class  $KL^\varphi[0, 1]$  is closed under the operator  $C$ .

(v)  $C : L^\varphi(I) \rightarrow L^\varphi(I)$ .

We have the following implications: (i)  $\Rightarrow$  (iii)  $\Rightarrow$  (v) and (ii)  $\Rightarrow$  (iv)  $\Rightarrow$  (v).

*Proof.* The proof is similar to the proof of Proposition 7 and 9.  $\square$

**Remark 11.**

- (i) Condition (iv) in Proposition 8 and condition (v) in Proposition 9 is equivalent to  $Ces_\varphi[0, \infty) \neq \{0\}$ . This follows from Theorem 1 (a) in [26], see also Proposition 3 in [21].
- (ii) Recall that the condition (ii) from Propositions 7, 8 and condition (iii) from Propositions 9, 10 have been applied to prove the existence of order isomorphically isometric copy of  $l^\infty$  in  $Ces_\varphi[0, 1]$  - see Theorems 3, 4.  
Notice also that condition  $C : L^\varphi \rightarrow L^\varphi$  has been used in [21] to prove the criteria for order continuity of  $Ces_\varphi$  (equivalently for the existence of isomorphic copy of  $l^\infty$  in  $Ces_\varphi$ , see Theorem A).

A function  $f : \mathbb{R}_+ \rightarrow \mathbb{R}_+$  is said to be pseudo-increasing for all arguments (for small argument or large arguments) whenever there exist constant  $M > 0$ ,  $u_0 \geq 0$  with  $f(u) \leq Mf(v)$  for all  $0 \leq u < v$  ( $0 \leq u < v \leq u_0$  or  $u_0 \leq u < v$ , respectively). The following useful characterization is well known.

**Lemma 12.** Assume  $\varphi$  is an Orlicz function and  $p > 1$ . Function  $\varphi(u)^{1/p}/u$  is pseudo-increasing (for small arguments, large arguments or all arguments) if and only if  $\varphi^{1/p}$  is equivalent to convex function  $\gamma$  (for small arguments, large arguments or all arguments respectively).

*Proof.* ( $\Rightarrow$ ). See [20, Theorem 1.6].

( $\Leftarrow$ ). We were not able to find that simple proof, so we present it for the reader's convenience. Note that if  $\gamma$  is convex function then for all  $t < s$

$$\gamma(s) = \gamma\left(\frac{s}{t}t\right) \geq \frac{s}{t}\gamma(t),$$

therefore

$$\frac{\gamma(t)}{t} \leq \frac{\gamma(s)}{s}.$$

Suppose  $\varphi^{1/p}$  is equivalent to convex function  $\gamma$  for all arguments, i.e. there exists constants  $A, B > 0$  such that for all  $u \in [0, \infty)$

$$A \frac{\gamma(u)}{u} \leq \frac{\varphi(u)^{1/p}}{u} \leq B \frac{\gamma(u)}{u}.$$

For  $u < v$  we have

$$\frac{\varphi(u)^{1/p}}{u} \leq B \frac{\gamma(u)}{u} \leq B \frac{\gamma(v)}{v} \leq \frac{B}{A} \frac{\varphi(v)^{1/p}}{v}.$$

$\square$

Applying Theorem 3 and 4, Proposition 7, 8, 9 and 10 we get following corollary.

**Corollary 13.** Suppose in the case when  $\varphi(b_\varphi) = \infty$  that there is  $p > 1$  such that the function  $\varphi^{1/p}$  is equivalent to a convex function for large arguments if  $I = [0, 1]$  or for all arguments if  $I = [0, \infty)$  (as we mean in the Proposition 7, 8, 9 and 10). If  $\varphi \notin \Delta_2$  then the corresponding Cesàro function space  $Ces_\varphi(I)$  contains an order isomorphically isometric copy of  $l^\infty$ .

**Example 14.** Let  $\varphi(u) = e^u - 1$  for  $u \geq 0$ . Then  $\varphi \notin \Delta_2(\infty)$  and for all  $p > 1$  the function  $\psi = \varphi^{1/p}$  is equivalent to a convex function for large arguments. In fact,

$$\limsup_{u \rightarrow \infty} \frac{e^{2u} - 1}{e^u - 1} = \infty,$$

and

$$\psi''(u) = \frac{1}{p^2} e^u (e^u - 1)^{\frac{1}{p}-2} (e^u - p) > 0$$

for all  $p > 1$  and  $u > \ln p$ .

**Example 15.** It may happen that  $\varphi \notin \Delta_2(\infty)$  and for any  $p > 1$  function  $\varphi^{1/p}$  is not equivalent to any convex function for large arguments. Indeed, take an Orlicz function  $\varphi$  with  $\varphi \notin \Delta_2(\infty)$  and  $\varphi^* \notin \Delta_2(\infty)$  (see for example [24, p. 28]). Since  $\varphi^* \notin \Delta_2(\infty)$  so  $\alpha_\varphi = 1$  (see [32]). Consequently,  $C$  is not bounded. Now we apply Proposition 7.

## ACKNOWLEDGEMENTS

The second author (Paweł Kolwicz) is supported by the Ministry of Science and Higher Education of Poland, grant number 04/43/DSPB/0089.

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